Engineering Notes

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Misstatements of the Test for Positive Semidefinite Matrices

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I. Introduction

A MISSTATEMENT of the principal minor test for symmetric matrices (as first identified in Ref. 1, rediscovered 11 years later in Ref. 2, and further clarified in Ref. 3) has been propagated in at least seven significant modern control textbooks since 1968. A recent control theory textbook[4] also makes this same mistake along two different lines by first stating (p. 468, Note 11.2) that "a well-known criterion for a matrix to be positive semidefinite is that its determinant and the determinants of all its principal minors be nonnegative." Along the almost trivial first line (as already addressed in Ref. 5), such a test can only be applied to symmetric matrices or to the symmetric representation of the original matrix as it appears in arbitrary vector inner products, rather than to general square matrices. Along the more important second line, there are transparent counterexamples that demonstrate that just considering principal minors to confirm positive semidefiniteness does not suffice.[1,2]

The analytic pitfall that all of these textbooks have stumbled into is in interpreting the test for positive semidefiniteness in too strong an analogy with the valid principal minor test for positive definiteness (also referred to as Sylvester's criterion), which requires only that all leading principal minors be positive in order to conclude that the $n \times n$ symmetric matrix is in fact positive definite. From Ref. 2 (p. 122, Eq. 3), for positive *semidefiniteness,* all $2^n - 1$ possible subminors and not just the leading principal minors need to be considered in order to have a valid test. In Ref. 3, for balance, several textbooks are identified that have a correct statement of the test for positive semidefiniteness. However, as discussed in Ref. 3, for practical problems of realistically higher dimensions, the evaluation of multiple minors or determinants as offered in these correct textbooks is not a computationally efficient approach for determining whether a matrix is positive definite, negative definite, semidefinite, or indefinite. A preferred approach is to make use of the singular value decomposition (SVD) in making such a determination, as explained in Ref. 3. SVD was first observed in Ref. 6 to be the only computationally reliable method for establishing the rank of a matrix. There is a refinement of SVD, known as Aasen's method, that exploits underlying symmetry of the matrices (and only requires on the order of $n^3/6$ operations, where n is the dimension of the square matrix under test), and it is already available for these types of problems, as discussed in Ref. 7 (pp. 101–106).

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II. Counterexample to the Stated Partitioned Criterion

In Ref. 4 (p. 468, Note 11.2), a mistake is now made along yet a third line by going on to say that the partitioned matrix

$$Z = \begin{bmatrix} V & X \\ X^T & W \end{bmatrix} \tag{1}$$

is positive semidefinite if and only if each of the following determinants is nonnegative as

$$|V| \geq 0, \ |W| \geq 0, \qquad \det \begin{bmatrix} V & X \\ X^T & W \end{bmatrix} \geq 0 \tag{2}$$

Unfortunately, the preceding claim is violated even for the case of W being a (1×1) matrix or scalar, as can be demonstrated by considering the 3×3 counterexample (offered in Ref. 1 in another context) as

$$Z_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \tag{3}$$

when partitioned as

$$W = [0], \qquad X^T = [1 \quad 1], \qquad V = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{4}$$

satisfying all the conditions of Eq. (2), yet the matrix Z_1 in Eq. (3) has eigenvalues of 0, $1 + \sqrt{3}$, $1 - \sqrt{3}$ (the last eigenvalue being clearly negative) so that Z_1 cannot be positive semidefinite despite its successfully passing all of the tests of Ref. 4 (p. 468, Note 11.2).

A better partitioned determinantal criterion to use, which does not involve any nontenable assumptions on W^{-1} existing [as invoked in the "proof" offered in Ref. 4 (p. 468, Note 11.2)] is as stated in Ref. 8 (p. 745, Property 1) where if a symmetric matrix P is partitioned as

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \tag{5}$$

then

$$P \geq 0 \tag{6}$$

if and only if

$$P_3 \geq 0; \qquad P_1 - P_2 P_3^{\dagger} P_2^T \geq 0 \tag{7}$$

and

$$\mathfrak{N}[P_3] \subset \mathfrak{N}[P_2] \tag{8}$$

where

$$P_3^{\dagger} = \text{Moore-Penrose pseudoinverse of } P_3$$

and

$$\mathfrak{N}[P_i] = \text{null space of } P_i \text{ (see Ref. 9)}$$

Proof of the necessity and sufficiency between the condition of Eq. (6) and that of Eqs. (7) and (8) is provided in Appendix I of Ref. 8. Notice that the condition of Eqs. (7) and (8) differs from the implicit assumption invoked in Ref. 4 (Note 11.2) (that W is nonsingular) in three respects. First, P_3^{-1} is not assumed to exist; use of the pseudoinverse $P_3^{\dagger}$ will properly handle any situation regarding P_3, whether or not zero eigenvalues are present. Second, the condition of Eq. (7) is such that the submatrices are established to be positive semidefinite rather than just a condition on the associated determinants, as in Eq. 2. Third, there is an additional criterion present on the nesting of null spaces[9] of P_3 and P_2; the condition of Eq. (8) must also be satisfied before the conclusion can be made that the P of Eq. (5) is positive semidefinite.

The condition of Eq. (8) fails to be satisfied for the numerical example of Eq. (4) since it reduces to

$$R \not\subset \mathcal{N}[P_2] \equiv \phi \tag{9}$$

thus enabling the correct conclusion to be drawn that the matrix of Eq. (3) is *not* positive semidefinite. When the condition of Eq. (8) is applied to the second example of Ref. 2 (Eq. 2) [identical to Eq. (3) here except that the parameter value a appears in the place of the zero in the third row and column], then the resulting

$$\phi \subset \mathcal{N}[P_2] \tag{10}$$

satisfies the required *nested subspace property* (for all nonzero values of a) and enables the correct conclusions to be drawn that the corresponding full matrix P is indeed positive semidefinite for $a \geq 1$, while failing to be so if $1 > a > 0$ [where the second condition of Eq. (7) correctly comes into play to reveal this lack of positive semidefiniteness in the latter case of $1 > a$].

III. Conclusion

Some prevalent misconceptions on how to test matrices for positive semidefiniteness (both theoretically and computationally) were reviewed. A simple counterexample revealed that a recently offered partitioned test for demonstrating the positive semidefiniteness of a matrix (with the potential of being applied stagewise to the higher dimensional matrices encountered in industrial applications) is flawed. A proper version of such a test was discovered, as historically developed by others in preparing to perform matrix spectral factorization (which involves matrices whose entries are polynomials or rational functions of a complex variable), but which is also valid in the simpler case here where the matrices of interest have constant numerical entries. The key difference between the incorrect and correct version of the partitioned test is that a condition involving the nesting of associated null spaces corresponding to two of the critical partitions must also be satisfied in order to properly conclude that the matrix is positive semidefinite.

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Estimating Projections of the Controllable Set

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Introduction

THE theory used to find controllable or reachable sets provides a useful tool in engineering design and analysis. For example, consider the following illustrative problem: A simple spring mass (k-m) system subject to a control force f is described by

$$m\ddot{y} + ky = f \tag{1}$$

where y is the displacement of the mass. Suppose that $k/m = 6/s^2$ and that a proportional, integral, derivative (PID) control loop of the form

$$f = m[r - (5y + 6\int y\,dt + 6\dot{y})] \tag{2}$$

is put about the system so that the controlled dynamics are given by

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = r \tag{3}$$

where r is a command input. The problem is to determine the maximum possible energy in the spring-mass system if it is initially at equilibrium, but for time $t > 0$, it is subject to an unknown but bounded command input. This problem may be solved by first finding those points in the three-dimensional state space (position-velocity-acceleration) that are reachable from the equilibrium point under the bounded command input (the reachable set) and then finding where on the reachable set the energy ($\frac{1}{2}m\dot{y}^2 + \frac{1}{2}ky^2$) is maximized. Note, however, that since only the energy is of interest, it would be sufficient to have knowledge of only the projection of the reachable set onto the position-velocity space. In this case, as indeed with many other similar problems, it would be useful to have a theory that could directly provide the projected controllable or reachable set.

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